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SOLUTION OF THE PROBLEM OF STRESS CONCENTRATION AROUND INTERSECTING DEFECTS BY USING THE RIEMANN PROBLEM WITH AN INFINITE INDEX*

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An analytic method is proposed for solving discontinuous boundary value problems for harmonic and biharmonic operators based on relying on apparatus developed to solve the Riemann boundary value problem with an infinite index. Boundary value problems for differential equations are first reduced to a system of two singular integral equations (SIE) with a fixed singularity by the generalized method of integral transforms, and then to a certain Riemann problem with zero index on a contour parallel to the imaginary axis. Subsequent transformations reduce this problem to two successively solvable scalar Riemann problems (the first with a plus-infinity index and the second with a minus-infinity index). The first problem is solved to entire-function accuracy, found from the condition for the second problem to be solvable (a convolution type Fredholm integral equation on a segment). This method is applied to the solution of the antiplane problem for a plane with a Γ-shaped slit, and also to the plane problem for a plane containing a cruciform slit (the slit edges are free of tangential loads), where the slit branches are of different length in both problems. The singularity in the solution of the SIE system at zero (the intersection of the slit branches) and also the stress intensity coefficients are found. The final formulas are reduced to a form convenient for numerical realization.

Earlier /1/ SIE systems analogous to that under consideration were solved by approximate methods without taking account of the presence of the fixed singularity in the kernel.

1. Reduction of the antiplane problem for a plane with a Γ-shaped slit to a vector Riemann problem. The following boundary value problem of mathematical physics is considered for a plane with a slit:

$$\Delta u(x, y) = 0,\ (x, y) \in R^2 \setminus I \tag{1.1}$$

$$\frac{\partial u}{\partial y}(x, \pm 0) = q(x), x \in I_a;\quad \frac{\partial u}{\partial x}(\pm 0, y) = r(y),\ y \in I_b \tag{1.2}$$

$$\frac{\partial u}{\partial x}(x, -0) - \frac{\partial u}{\partial x}(x, +0) = \chi_1(x),\quad -\infty < x < +\infty \tag{1.3}$$

$$\frac{\partial u}{\partial y}(-0, y) - \frac{\partial u}{\partial y}(+0, y) = \chi_2(y),\quad -\infty < y < +\infty \tag{1.4}$$

$$I = \{(x,\ y) \in R^2{:}\ 0 < x < a,\ y = \pm 0;\ x = \pm 0,\ 0 < y < b\}$$

$$I_a = [0,\ a],\ I_b = [0,\ b];\ \chi_1(x) = 0,\ x \,\overline{\in}\, I_a;\ \chi_2(y) = 0,\ y \,\overline{\in}\, I_b$$

The functions $q(x)$ and $r(y)$ are known and satisfy the Hölder condition (from the classes $H(I_a)$ and $H(I_b)$, respectively). We understand the solution of the problem to be the set of functions $\{u(x, y), \chi_1(x), \chi_2(y)\}$ belonging to the following classes:

$$u(x, y) \in C^2(R^2 \setminus I),\quad \frac{\partial u}{\partial y}(x, 0) \in H(I_a),\quad \frac{\partial u}{\partial x}(0, y) \in H(I_b)$$

$$\frac{\partial u}{\partial x}(x, \pm 0) \in H^*(I_a),\quad \frac{\partial u}{\partial y}(\pm 0, y) \in H^*(I_b)$$

$$\chi_1(x) \in H^*(I_a),\ \chi_2(y) \in H^*(I_b)$$

satisfy relationships (1.1)-(1.4) as well as the closedness condition for the slit

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$$\int_0^a \chi_1(\xi)\,d\xi + \int_0^b \chi_2(\xi)\,d\xi = 0 \tag{1.5}$$

We note that $H^*(J)$ is a class of functions satisfying the Hölder condition everywhere with the segment J and allowing an integrable infinity at its ends.

The boundary value problem formulated describes /2/ the longitudinal shear of a plane with a slit I, where $u(x, y)$ are the longitudinal displacements of points of the plane.

Application of a generalized scheme /2/ of the integral transformation method (Fourier transforms) enables us to obtain

$$u(x,y) = -\frac{1}{2\pi}\int_0^a \chi_1(\xi)\,\mathrm{arctg}\,\frac{x-\xi}{y}\,d\xi + \frac{1}{2\pi}\int_0^b \chi_2(\xi)\,\mathrm{arctg}\,\frac{x}{y-\xi}\,d\xi + C \tag{1.6}$$

from (1.1), (1.3) and (1.4).

To determine the χ_1, χ_2 realizing condition (1.2) we arrive at a system of two SIE with a fixed singularity in the form

$$\frac{1}{\pi}\int_0^1 \frac{\omega_1(\tau)}{\tau-t}\,d\tau + \frac{1}{\pi}\int_0^1 \omega_2(\tau)\frac{\mu t\,d\tau}{\tau^2+\mu^2t^2} = h_1(t) \tag{1.7}$$

$$\frac{1}{\pi}\int_0^1 \omega_1(\tau)\frac{\mu^{-1}t\,d\tau}{\tau^2+\mu^{-2}t^2} + \frac{1}{\pi}\int_0^1 \frac{\omega_2(\tau)}{\tau-t}\,d\tau = h_2(t) \quad (0<t<1)$$

$$\omega_1(\tau) = \chi_1(a\tau),\ \omega_2(\tau) = \chi_2(b\tau)$$

$$h_1(t) = -2q(at),\ h_2(t) = -2r(bt),\ \mu = a/b$$

For $\mu = 1$ this system can be solved exactly if we introduce a pair of functions $\omega_\pm(\tau) = \omega_1(\tau) \pm \omega_2(\tau)$. Then the system reduces to two sequentially solvable equations of the Mellin convolution type. Their solution is constructed by the method described in /3/. We henceforth exclude the case $\mu = 1$ from consideration.

Let us predetermine the system (1.7) in the semi-infinite segment $1 < t < \infty$ by introducing the one-sided functions $\omega_{j\pm}(t)$ and $h_{j-}(t)$ $(j = 1, 2)$ such that $\omega_{j+}(t) = 0,\ 0 < t < 1$ and

$$\omega_{j-}(t) = \begin{cases}\omega_j(t), & 0<t<1\\ 0, & t>1\end{cases};\quad h_{j-}(t) = \begin{cases}h_j(t), & 0<t<1\\ 0, & t>1\end{cases}$$

Then

$$\frac{1}{\pi}\int_0^\infty \frac{\omega_{1-}(\tau)}{1-t/\tau}\frac{d\tau}{\tau} + \frac{1}{\pi}\int_0^\infty \frac{\omega_{2-}(\tau)\mu t/\tau}{1+\mu^2(t/\tau)^2}\frac{d\tau}{\tau} = h_{1-}(t) + \omega_{1+}(t) \tag{1.8}$$

$$\frac{1}{\pi}\int_0^\infty \frac{\omega_{1-}(\tau)\mu^{-1}t/\tau}{1+\mu^{-2}(t/\tau)^2}\frac{d\tau}{\tau} + \frac{1}{\pi}\int_0^\infty \frac{\omega_{2-}(\tau)}{1-t/\tau}\frac{d\tau}{\tau} = h_{2-}(t) + \omega_{2+}(t)$$

$$(0 < t < \infty)$$

Taking into account that the functions $h_j(t)$ are bounded as $t \to +0$ and $t \to 1-0$, and analysing the Cauchy type integrals in (1.8), we obtain

$$\omega_{j-}(t) = O(t^\alpha),\ t\to+0;\ \omega_{j-}(t) = O([1-t]^{-1/2}),\ t\to 1-0 \tag{1.9}$$

$$\omega_{j+}(t) = O([t-1]^{-1/2}),\ t\to 1+0;\ \omega_{j+}(t) = O(t^{-1})$$

$$t \to +\infty$$

$(|\alpha| < 1)$ and then the Mellin transforms

$$\varphi_j^+(s) = \int_1^\infty \omega_{j+}(t)t^s\,dt,\quad \varphi_j^-(s) = \int_0^1 \omega_{j-}(t)t^s\,dt \tag{1.10}$$

are analytic functions in the half-planes $\mathrm{Re}\,s < 0$ and $\mathrm{Re}\,s > -1-\alpha$, respectively. Now, if we take account of the values of the integrals

$$\int_0^\infty \frac{x^s\,ds}{1-x} = \pi\,\mathrm{ctg}\,\pi s,\quad \int_0^\infty \frac{\mu x^{s+1}\,dx}{1+\mu^2x^2} = -\frac{\pi\mu^{-s-1}}{2\sin \pi s/2}$$

that converge for $-1 < \operatorname{Re} s < 0$ and $-2 < \operatorname{Re} s < 0$, respectively, and use the notation

$$h_j^-(s) = \int_0^1 h_{j-}(t)\, t^s\, dt$$

($h_j^-(s)$ are analytic functions for $\operatorname{Re} s > -1$), then by using the Mellin convolution theorem /4/, we obtain the vector Riemann problem from (1.8)

$$\begin{aligned}
&\varphi^+(t) = G(t)\,\varphi^-(t) + h^-(t),\quad t \in L \\
&L = \{s : \operatorname{Re} s = \gamma^\circ\},\ \max\{-\alpha - 1,\ -1\} < \gamma^\circ < 0 \\
&G(s) = \begin{Vmatrix} \operatorname{ctg} \pi s & -\mu^{-s-1}(2\sin \pi s/2)^{-1} \\ -\mu^{s+1}(2\sin \pi s/2)^{-1} & \operatorname{ctg} \pi s \end{Vmatrix} \\
&(\Delta(s) = \det G(s) = (2\cos^2 \pi s - \cos \pi s - 1)\,(2\sin^2 \pi s)^{-1})
\end{aligned} \tag{1.11}$$

$\varphi(s) = \| \varphi_1(s),\ \varphi_2(s) \|$ is a piecewise-analytic vector with jump lines L, i.e., the vector $\varphi^+(s)$ is analytic in D^+ ($\operatorname{Re} s \leqslant \gamma^\circ$) and $\varphi^-(s)$ in D^- ($\operatorname{Re} s \geqslant \gamma^\circ$); $h^-(s) = -\| h_1^-(s),\ h_2^-(s) \|$ is an analytic vector in D^-.

Since $\Delta(-{}^2/_3) = 0$ problem (1.11) is examined on the following contour: $L = \{s : \operatorname{Re} s = \gamma^\circ\}$, $\max\{-\alpha - 1,\ -{}^2/_3\} < \gamma^\circ < 0$. Then $\Delta(t) \neq 0$ anywhere on L. Taking into account that $\Delta(\gamma^\circ + i\tau) \to -1$, $\tau \to \pm\infty$ and $\Delta(\gamma^\circ) < 0$ ($\Delta(\gamma^\circ) \to -{}^3/_4$, $\gamma^\circ \to -0$), we find that the total index of problem (1.11) equals zero.

By virtue of (1.5) the solution of problem (1.11) is sought which satifies the condition

$$\mu\varphi_1^-(0) + \varphi_2^-(0) = 0 \tag{1.12}$$

Taking (1.9) and (1.10) into account as well as an Abelian type theorem for the Laplace transorm (/5/, p.473), we find the behaviour of the Mellin transform $\varphi_j^\pm(s)$ at infinity. We consequently find $\varphi(s) = O(s^{-1/2})$, $s \to \infty$, $s \in L$.

2. Solution of the vector Riemann problem. We will henceforth consider $\mu > 1$ everywhere without loss of generality. We represent the matrix $G(s)$ in the form $G(s) = G_+(s)\,G_0(s)G_-(s)$, where

$$\begin{aligned}
&G_+(s) = \begin{Vmatrix} 0 & 1 \\ 1 & \mu^{s+1} \end{Vmatrix},\quad G_-(s) = \begin{Vmatrix} 1 & -\mu^{-s-1} \\ 0 & 1 \end{Vmatrix} \\
&G_0(s) = \begin{Vmatrix} d_1\mu^{s+1} & 0 \\ \operatorname{ctg} \pi s & d_2\mu^{-s-1} \end{Vmatrix},\quad \begin{aligned} &d_j = -(2\sin \pi s/2)^{-1} + (-1)^j \operatorname{ctg} \pi s \\ &\det G_\pm(s) = \mp 1 \end{aligned}
\end{aligned} \tag{2.1}$$

The elements of the matrix $G_\pm(s)$ are analytic functions in $D^\pm$. We factorize the expressions d_j by using Γ-functions

$$\begin{aligned}
&d_j(s) = K_j^+(s)\,[K_j^-(s)]^{-1}\ (j = 1,\ 2) \\
&K_1^\pm(s) = \left[\frac{\Gamma({}^1/_2 \mp {}^1/_2 \mp s/4)\,\Gamma({}^1/_4 \mp s/4)\,\Gamma({}^3/_4 \mp s/4)}{\Gamma({}^1/_6 \mp s/4)\,\Gamma({}^1/_2 \mp s/4)\,\Gamma({}^5/_6 \mp s/4)} \right]^{\pm 1} \\
&K_2^\pm(s) = \left[\frac{\Gamma({}^1/_2 \mp s/4)\,\Gamma({}^1/_4 \mp s/4)\,\Gamma({}^3/_4 \mp s/4)}{\Gamma({}^1/_2 \mp {}^1/_2 \mp s/4)\,\Gamma({}^1/_3 \mp s/4)\,\Gamma({}^2/_3 \mp s/4)} \right]^{\pm 1}
\end{aligned} \tag{2.2}$$

The functions $K_j^\pm(s)$ are analytic and have no zeros in $D^\pm$, where

$$K_1^\pm(s) \sim (\mp s/4)^{-1/2},\quad K_2^\pm(s) \sim (\mp s/4)^{1/2},\quad s \to \infty,\ s \in D^\pm \tag{2.3}$$

Taking account of (2.1) and (2.2) and introducing the piecewise-analytic vector

$$\begin{aligned}
&\Phi(s) = \| \Phi_1(s),\ \Phi_2(s) \| \\
&\Phi_1^+(s) = \frac{-\mu^{s+1}\varphi_1^+(s) + \varphi_2^+(s)}{K_1^+(s)},\quad \Phi_2^+(s) = \frac{\varphi_1^+(s)}{K_2^+(s)},\quad s \in D^+ \\
&\Phi_1^-(s) = \frac{\varphi_1^-(s) - \mu^{-s-1}\varphi_2^-(s)}{K_1^-(s)},\quad \Phi_2^-(s) = \frac{\varphi_2^-(s)}{K_2^-(s)},\quad s \in D^-
\end{aligned} \tag{2.4}$$

into consideration, we obtain that the vector problem (1.11) is equivalent to two successively solvable Riemann problems

$$\begin{aligned}
&1^\circ.\ \Phi_1^+(t) = \mu^{t+1}\,\Phi_1^-(t) + g_1(t),\quad t \in L \\
&g_1(t) = [\mu^{t+1}h_1^-(t) - h_2^-(t)]\,[K_1^+(t)]^{-1}
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&2^\circ.\ \Phi_2^+(t) = \mu^{-t-1}\Phi_2^-(t) + g_2(t),\quad t \in L \\
&g_2(t) = [\operatorname{ctg} \pi t K_1^-(t)\,\Phi_1^-(t) - h_1^-(t)]\,[K_2^+(t)]^{-1}
\end{aligned} \tag{2.6}$$

where $\Phi_1(s) = O(1)$, $\Phi_2(s) = O(s^{-1})$, $s \to \infty$, $s \in L$; $g_1(t) = O(t^{-1/2})$, $g_2(t) = O(t^{-1})$ for $t \to \infty$, $t \in L$.

The set of all functions analytic in $D^{\pm}$ and decreasing as s^{-1} in $D^{\pm}$ as $s \to \infty$, will be denoted by $M_0^{\pm}$ and the bounded functions by $M_1^{\pm}$.

The solution of problem 1°, with the plus-infinity index ($\mu > 1$), is sought in the class $M_1^{\pm}$ and the solution of problem 2°, with the minus infinity index, is sought in $M_0^{\pm}$. The solution of problem 1° and 2° is determined by the following theorems ($g(t)$ is a function satisfying the condition $g(t) = O(t^{\varkappa})$, $t \to \infty$, $t \in L$, $\varkappa < 0$).

Theorem 1. The inhomogeneous Riemann problem

$$\Phi^+(t) = \nu^{t+1}\Phi^-(t) + g(t), t \in L \tag{2.7}$$

with the plus-infinity index ($\nu > 1$) has an infinite number of solutions in the class $M_j^{\pm}$ ($j = 0, 1$)

$$\Phi^+(s) = \delta_{j1}(C_1 + C_2\nu^{s+1}) + \nu^{s+1}P(s) + \int_1^{\infty} g^{\circ}(x)\, x^s\, dx, \quad s \in D^+ \tag{2.8}$$

$$\Phi^-(s) = \delta_{j1}(C_1\nu^{-s-1} + C_2) + P(s) - \frac{1}{\nu^{s+1}}\int_0^1 g^{\circ}(x)\, x^s\, dx, \quad s \in D^-$$

$$P(s) = \int_{1/\nu}^{1} \gamma(x)\, x^s\, dx, \quad g^{\circ}(x) = \frac{1}{2\pi i}\int_L g(s)\, x^{-s-1}\, ds$$

where C_1, C_2 are arbitrary constants, δ_{j1} is the Kronecker delta, and $\gamma(x)$ is an arbitrary function from the space $L_1(1/\nu, 1)$.

Theorem 2. The homogeneous ($g(t) \equiv 0$) Riemann problem (2.7) with minus-infinity index ($0 < \nu < 1$) has no solutions in the class $M_j^{\pm}$. It is necessary and sufficient for the solution of the inhomogeneous problem that $g^{\circ}(x) = 0$ for $\nu \leqslant x \leqslant 1$. The unique solution of (2.7) for this condition to be satisfied has the form

$$\Phi^+(s) = \int_1^{\infty} g^{\circ}(x)\, x^s\, dx, \quad s \in D^+; \quad \Phi^-(s) = -\frac{1}{\nu^{s+1}}\int_0^{\nu} g^{\circ}(x)\, x^s\, dx, \quad s \in D^- \tag{2.9}$$

Proof of Theorem 1. Let $\Phi^{\pm}(s) \in M_1^{\pm}$, then there exists one and only one pair of functions $\varphi_+(x) \in L_1(1, \infty)$, $\varphi_-(x) \in L_1(0, 1)$ bounded as $x \to 1 + 0$, $x \to 1 - 0$, respectively, such that

$$\Phi^+(s) = \int_1^{\infty} \varphi_+(x)\, x^s\, dx, \quad \Phi^-(s) = \int_0^1 \varphi_-(x)\, x^s\, dx \tag{2.10}$$

Taking into account that $g(t) = O(t^{\varkappa})$, $t \to \infty$, $t \in L$ ($\varkappa < 0$), we have

$$g(t) = \int_0^{\infty} g^{\circ}(x)\, x^t\, dx, \quad g^{\circ}(x) \in L_1(0, \infty)$$

Let us predefine: $\varphi_+(x) = 0$, $0 < x < 1$ and $\varphi_-(x) = 0$, $x > 1$, consequently, we obtain instead of the boundary value problem (2.7)

$$\varphi_+(x) = \varphi_-(x/\nu) + g^{\circ}(x), \quad 0 < x < \infty \tag{2.11}$$
$$\varphi_+(x) = 0, \quad 0 < x < 1; \quad \varphi_-(x) = 0, \quad x > 1$$

We consider three cases

a) $0 < x < 1$, $0 = \varphi_-(x/\nu) + g^{\circ}(x)$;
b) $1 \leqslant x \leqslant \nu$, $\varphi_+(x) = \varphi_-(x/\nu) + g^{\circ}(x)$;
c) $x > \nu$, $\varphi_+(x) = g^{\circ}(x)$.

By using the inverse transform we hence find

$$\Phi^+(s) = \nu^{s+1}\int_{1/\nu}^{1} \varphi_-(x)\, x^s\, dx + \int_1^{\infty} g^{\circ}(x)\, x^s\, dx, \quad s \in D^+ \tag{2.12}$$

$$\Phi^-(s) = \int_{1/\nu}^{1} \varphi_-(x)\, x^s\, dx - \frac{1}{\nu^{s+1}}\int_0^1 g^{\circ}(x)\, x^s\, dx, \quad s \in D^-$$

where $\varphi_-(x)$ is an arbitrary function from $L_1(1/\nu, 1)$ so that (2.8) is proved in the case $j = 0$.

Now let $\Phi^{\pm}(s) \in M_1^{\pm}$. Then $\varphi_+(x)$ and $\varphi_-(x)$ are generalized functions in the representation (2.10). Following the scheme of the proof for the case $j = 0$, we arrive at formulas (2.12) where

$$\varphi_-(x) = \lim_{\varepsilon_1, \varepsilon_2 \to +0} \varphi_-^{\varepsilon_1, \varepsilon_2}(x)$$

$$\varphi_-^{\varepsilon_1, \varepsilon_2}(x) = \begin{cases} \varepsilon_1 [\varepsilon_1^2 + (x - 1/\nu)^2]^{-1} C_1^\circ, & 1/\nu < x < 1/\nu + \varepsilon_1 \\ \gamma(x), & 1/\nu + \varepsilon_1 < x < 1 - \varepsilon_2 \\ \varepsilon_2 [\varepsilon_2^2 + (1 - x)^2]^{-1} C_2^\circ, & 1 - \varepsilon_2 < x < 1 \end{cases}$$

(C_1°, C_2° are arbitrary constants), and then taking into account that

$$P(s) = \lim_{\varepsilon_1, \varepsilon_2 \to +0} \int_{1/\nu}^{1} \varphi_-^{\varepsilon_1, \varepsilon_2}(x) x^s \, dx = \frac{\pi}{2} (C_1^\circ \nu^{-s-1} + C_2^\circ) + \int_{1/\nu}^{1} \gamma(x) x^s \, dx$$

we arrive at (2.8) for $j = 1$.

Proof of Theorem 2. Using the same reasoning as for the proof of Theorem 1, we arrive at problem (2.11) for $0 < \nu < 1$:

a) $0 < x < \nu$, $0 = \varphi_-(x/\nu) + g^\circ(x)$;

b) $\nu \leqslant x \leqslant 1$, $0 = g^\circ(x)$;

c) $x > 1$, $\varphi_+(x) = g^\circ(x.)$

We hence obtain the solvability condition for problem (2.7) and formula (2.9).

Similar theorems governing the solution (in other spaces) of the Riemann problem with an infinite index $\Phi^+(x) = e^{i\alpha x}\Phi^-(x) + g(x)$, $-\infty < x < +\infty$, are presented in /6/.

We turn to problems 1° and 2°. For the solvability conditions

$$g_2^\circ(x) = 0, \quad 1/\mu \leqslant x \leqslant 1 \tag{2.13}$$

to be satisfied the solutions of problems (2.5) have the form

$$\Phi_1^+(s) = C_1 + C_2\mu^{s+1} + \mu^{s+1}P(s) + \int_1^\infty g_1^\circ(x) x^s \, dx, \quad s \in D^+ \tag{2.14}$$

$$\Phi_1^-(s) = C_1\mu^{-s-1} + C_2 + P(s) - \mu^{-s-1}\int_0^1 g_1^\circ(x) x^s \, dx, \quad s \in D^-$$

$$\Phi_2^+(s) = \int_1^\infty g_2^\circ(x) x^s \, dx, \quad s \in D^+;$$

$$\Phi_2^-(s) = -\mu^{s+1} \int_0^{1/\mu} g_2^\circ(x) x^s \, dx, \quad s \in D^-$$

$$P(s) = \int_{1/\mu}^{1} \gamma(\xi) \xi^s \, d\xi, \quad g_j^\circ(x) = \frac{1}{2\pi i} \int_L g_j(s) x^{-s-1} \, ds \quad (j = 1, 2)$$

by virtue of Theorems 1 and 2.

We transform the solvability condition (2.13). Substituting the second relation into (2.14) into (2.6), we obtain an integral equation in the function $\gamma(\xi)$

$$\int_{1/\mu}^{1} \gamma(\xi) \Lambda\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi} = -C_2\Lambda(x) - C_1\Lambda(\mu x) + g_0(x) \tag{2.15}$$

$$1/\mu \leqslant x \leqslant 1$$

$$\Lambda(x) = -\frac{1}{2\pi i} \int_L \frac{K_1^\pm(s)}{K_2^\pm(s)} \frac{\cos \pi s x^{-s-1} \, ds}{\cos \pi s/2 \pm \cos \pi s}, \quad x \lesseqgtr 1 \tag{2.16}$$

$$g_0(x) = \int_0^1 g_1^\circ(\xi) \Lambda\left(\frac{\mu x}{\xi}\right) \frac{d\xi}{\xi} + \frac{1}{2\pi i} \int_L \frac{h_1^-(s)}{K_2^+(s)} x^{-s-1} \, ds$$

In particular, when a constant load is applied to the slit edges and $q = r$, then $h_1(t) = h_2(t) = h = \mathrm{const}$ and

$$g_0(x) = [K_2^+(-1)]^{-1}h, \quad K_2^+(-1) = 3^{1/4}(2\sqrt{\pi})^{-1}$$

The integral (2.16) is evaluated by the theorem of residues. We have ($\psi(x)$ is the psi function)

$$\Lambda(x) = \frac{2}{3\pi\sqrt{3}} \sum_{m=0}^{\infty} \left[W_{m1}^\pm(x) + \frac{\sqrt{3}}{\pi}(\Psi_m^\pm - \ln x) W_{m2}^\pm(x) - W_{m3}^\pm(x)\right], \quad x \lesseqgtr 1$$

$$W_{mj}^\pm(x) = K_1^\pm(z_{mj}^\pm)[K_2^\pm(z_{mj}^\pm)]^{-1} x^{-z_{mj}^\pm - 1} \tag{2.17}$$

$$\Psi_m^{\pm} = 2\psi\left(\frac{1}{2} \mp \frac{z_{m1}^{\pm}}{4}\right) + \psi\left(\frac{1}{6} \mp \frac{z_{m2}^{\pm}}{4}\right) + \psi\left(\frac{5}{6} \mp \frac{z_{m2}^{\pm}}{4}\right) -$$
$$2\psi\left(\frac{1}{2} \mp \frac{1}{2} \mp \frac{z_{m1}^{\pm}}{4}\right) - \psi\left(\frac{1}{3} \mp \frac{z_{m1}^{\pm}}{4}\right) - \psi\left(\frac{2}{3} \mp \frac{z_{m2}^{\pm}}{4}\right)$$
$$z_{m1}^{+} = -{}^2/_3 - 4m, \quad z_{m2}^{+} = -2 - 4m, \quad z_{m3}^{+} = -10/3 - 4m$$
$$z_{m1}^{-} = {}^4/_3 + 4m, \quad z_{m2}^{-} = 4m, \quad z_{m3}^{-} = {}^8/_3 + 4m \quad (m = 0, 1, \ldots)$$

Letting $\Lambda_*(x)$ denote a continuous function $\Lambda_*(x) = \Lambda(x) - k_0 \operatorname{sgn}(\ln x)$ for $0 < x < \infty$, where $k_0 = [\Lambda(1+0) - \Lambda(1-0)]/2 \neq 0$, we obtain an equation in place of (2.15), with the kernel

$$\Lambda_*(x/\xi) + k_0 \operatorname{sgn}(\ln x/\xi)$$

After differentiation with respect to x this equation becomes a Fredholm integral equation of the second kind which will be equivalent to the initial equation (2.15) if compliance with (2.15) at the arbitrary point $x_0 \in [1/\mu, 1]$ is required. The equivalence condition fixes one of the arbitrary constants C_1, C_2. The other constant is determined from the condition that the slit (1.12) is closed, which will take the following form when the last two relationships in (2.4) are taken into account:

$$\sqrt{3}\pi\mu\Phi_1^-(0) + 2\Phi_2^-(0) = 0 \tag{2.18}$$

Taking into account that only the entire function $\mu^{s+1}P(s)$ must be known to construct the solution of problems (2.5), solving the integral Eq.(2.15) can be avoided. The following recipe is more convenient for a numerical realization. Let $h_j(t) \in C^1[0, 1]$ $(j = 1, 2)$, then the function $\gamma(\xi)$ allows of the decomposition

$$\gamma(\xi) = \sum_{k=-\infty}^{\infty} a_k \xi^{k-1/2}, \quad |a_k| < k^{-\alpha}, \quad |a_{-k}| < \mu^k k^{-\alpha}$$
$$(\alpha > 1,\ k = 1, 2, \ldots)$$

and we have for the entire function $\mu^{s+1}P(s)$

$$\mu^{s+1}P(s) = \sum_{k=-\infty}^{\infty} \frac{\mu^{s+1} - \mu^{-k+1/2}}{k + s + 1/2} a_k$$

We transform the solvability condition for (2.13) to the form

$$g_2^\circ(x) \equiv \sum_{k=-\infty}^{\infty} [M_k^+(x) - \mu^{-k+1/2} M_k^-(\mu x)] a_k + \tag{2.19}$$
$$C_2\Lambda(x) + C_1\Lambda(\mu x) - g_0(x) = 0, \quad 1/\mu \leqslant x \leqslant 1$$
$$M_k^{\pm}(x) = \frac{2}{3\pi\sqrt{3}} \sum_{m=0}^{\infty} \Big[V_{mk1}^{\pm}(x) + \frac{\sqrt{3}}{\pi}(\Psi_m^{\pm} - (z_{m2}^{\pm} + k + 1/2)^{-1} -$$
$$\ln x) V_{mk2}^{\pm}(x) - V_{mk3}^{\pm}(x)\Big], \quad x \gtrless 1$$
$$V_{mkj}^{\pm}(x) = (z_{mj}^{\pm} + k + 1/2)^{-1} W_{mj}^{\pm}(x)$$

(the notation of (2.17) is used here). The unknown coefficients a_k, C_1 are found from (2.19), say, by the collocation method, and the constant C_2 from condition (2.18), which we transform into

$$[1/2\sqrt{3}\pi + \Lambda^\circ(1)]C_1 + [1/2\sqrt{3}\pi\mu + \Lambda^\circ(\mu)]C_2 - 1/2\sqrt{3}\pi\theta_1 +$$
$$\theta_2 + \sum_{k=-\infty}^{\infty} a_k \Big[\sqrt{3}\pi\mu \frac{1 - \mu^{-k-1/2}}{2k+1} + M_k^\circ(\mu) - \frac{M_k^\circ(1)}{\mu^{k-1/2}}\Big] = 0$$
$$\theta_1 = \int_0^1 g_1^\circ(x)\,dx, \quad \theta_2 = \mu \int_0^{1/\mu} g_0(\ )\,dx$$
$$M_k^\circ(\mu) = \frac{2}{3\pi\sqrt{3}} \sum_{m=0}^{\infty} \Big[\frac{V_{mk1}^+(\mu^{-1})}{z_{m1}^+} + \frac{\sqrt{3}}{\pi}\Big(\Psi_m^+ - \frac{1}{z_{m2}^+} -$$
$$(z_{m2}^+ + k + 1/2)^{-1} + \ln\mu\Big)(z_{m2}^+)^{-1} V_{mk2}^+(\mu^{-1}) - (z_{m3}^+)^{-1} V_{mk3}^+(\mu^{-1})\Big]$$
$$\Lambda^\circ(\mu) = \frac{2}{3\pi\sqrt{3}} \sum_{m=0}^{\infty} \Big[\frac{W_{m1}^+(\mu^{-1})}{z_{m2}^+} + \frac{\sqrt{3}}{\pi}\Big(\Psi_m^+ - \frac{1}{Z_{m2}^+} + \ln\mu\Big) \times$$
$$(z_{m2}^+)^{-1} W_{m2}^+(\mu^{-1}) - (z_{m3}^+)^{-1} W_{m3}^+(\mu^{-1})\Big], \quad \mu \geqslant 1$$

The solution of the initial vector Riemann problem (1.11) is determined by the formulas

$$\varphi_1^-(s) = K_1^-(s)\,\Phi_1^-(s) + \mu^{-s-1}K_2^-(s)\,\Phi_2^-(s),\quad \varphi_1^+(s) = K_2^+(s)\,\Phi_2^+(s \qquad (2.20)$$
$$\varphi_2^-(s) = K_2^-(s)\,\Phi_2^-(s),\quad \varphi_2^+(s) = K_1^+(s)\,\Phi_1^+(s) + \mu^{s+1}K_2^+(s)\,\Phi_2^+(s)$$

3. **Determination of the singularity of the solution and the stress intensity coefficients.** Taking (2.20), (2.5), (2.6) and (2.2) into account, we find the solution of the SIE system (1.7) by using the inverse Mellin transform

$$\omega_j(\tau) = \frac{1}{2\pi i}\int_L \varphi_j^-(s)\,\tau^{-s-1}\,ds = -\frac{1}{\pi i}\int_L \left\{\frac{K_1^+(s)\,[\Phi_1^+(s) - g_1(s)]}{A_j\sin 3\pi s/2\sec \pi s} + \right.$$
$$\left.\frac{K_2^+(s)\,\Phi_2^+(s) + h_1^-(s)}{A_{3-j}^{-1}\sin 3\pi s/4\sec \pi s/4}\right\}\frac{\tau^{-s-1}\,ds}{\sec \pi s/2},\quad A_1 = \mu^{s+1},\ A_2 = 1$$

By using the theorem of residues we hence find the singularity of the solution

$$\omega_j(\tau) \sim a_j\tau^{-1/2},\ \tau \to +0\ (a_j = \text{const})$$

Let us introduce the stress intensity coefficients

$$N_a^{\pm} = \lim_{x\to a+0}\sqrt{2\pi(x-a)}\,\tau_{xz}(x, \pm 0),$$
$$N_b^{\pm} = \lim_{y\to b-0}\sqrt{2\pi(b-y)}\,\tau_{yz}(\pm 0, y)$$
$$(\tau_{xz} = G\partial u/\partial x,\ \tau_{yz} = G\partial u/\partial y)$$

where G is the shear modulus. Furthermore, following the scheme in /7/, we obtain the following relationships on the basis of (1.6):

$$N_a^+ = -N_a^- = -G\sqrt{\pi a/2}\,N_1,\ N_b^+ = -N_b^- = -G\sqrt{\pi b/2}\,N_2, \qquad (3.1)$$
$$N_j = \lim_{\xi\to 1-0}\sqrt{1-\xi}\,\omega_j(\xi)$$

We will determine the behaviour of the functions $\varphi_1^-(s)$, $\varphi_2^-(s)$ at infinity. Taking into account the relationships

$$\Phi_1^-(s) \sim C_2,\ \Phi_2^-(s) \sim -g_2^\circ(\mu^{-1})\,s^{-1},\ s\to\infty,\ s\in D^-$$

we have on the basis of (2.20) and (2.3)

$$\varphi_1^-(s) \sim 2C_2 s^{-1/2},\ \varphi_2^-(s) \sim -2^{-1}g_2^\circ(\mu^{-1})\,s^{-1/2},\ s\to\infty,\ s\in D^- \qquad (3.2)$$

On the other hand, according to theorems of Abelian type

$$\varphi_1^-(s) \sim N_1\sqrt{\pi}s^{-1/2},\quad \varphi_2^-(s) \sim N_2\sqrt{\pi}s^{-1/2},\ s\to\infty,\quad s\in D^-\setminus L \qquad (3.3)$$

Comparing (3.2) and (3.3) and taking account of (3.1), we obtain the following expressions for the unknown coefficients:

$$N_a^+ = -N_a^- = -G\sqrt{2a}C_2,\ N_b^+ = -N_b^- = G\sqrt{b}\,(2\sqrt{2})^{-1}g_2^\circ(\mu^{-1})$$

where $g_2^\circ(\mu^{-1})$ is found from (2.19).

4. **The plane problem for an elastic plane with a cruciform slit.** Let us consider a plane R^2 with a slit $J = \{(x, y)\in R^2\colon\ -a < x < a,\ y = \pm 0;\ x = \pm 0,\ -b < y < b\}$. A normal load $\sigma_y(x, \pm 0) = q(x),\ -a < x < a;\ \sigma_x(\pm 0,\ y) = r(y),\ -b < y < b\ (q(x) = q(-x),\ r(y) = r(-y))$ is applied to the slit edges and there is no tangential load. The normal displacements on the slit undergo the discontinuity

$$\left\langle\frac{\partial u}{\partial x}(x, 0)\right\rangle = \chi_1(x),\ |x| < \infty;\ \left\langle\frac{\partial u}{\partial y}(0, y)\right\rangle = \chi_2(y),\ |y| < \infty$$

where $\langle f(0)\rangle = f(-0) - f(+0)$; $\chi_1(x) = 0,\ x \overline{\in} [-a, a]$, $\chi_2(y) = 0,\ y \overline{\in} [-b, b]$. The tangential displacements are discontinuous.

This problem is equivalent to the following discontinuous boundary value problem for a biharmonic operator

$$\Delta^2 U(x, y) = 0,\quad (x, y)\in \mathbb{R}^2\setminus J \qquad (4.1)$$
$$\left\langle\frac{\partial^k U(x, 0)}{\partial y^k}\right\rangle = \left\langle\frac{\partial^k U(0, y)}{\partial x^k}\right\rangle = 0\quad (k = 0, 1, 2)$$
$$\left\langle\frac{\partial^3 U(x, 0)}{\partial y^3}\right\rangle = -\frac{\chi_1'(x)}{c},\quad \left\langle\frac{\partial^3 U(0, y)}{\partial x^3}\right\rangle = -\frac{\chi_2'(y)}{c},\quad c = \frac{1-\nu^2}{E}$$

($U(x, y)$ is the stress function, ν is Poisson's ratio, and E is the elastic modulus). The functions $\chi_1(x), \chi_2(y)$ should satisfy the condition that the slit is closed

$$\int_0^a \chi_1(x)\,dx + \int_0^b \chi_2(y)\,dy = 0$$

Using the discontinuous solution for an elastic plane /2/, p.212) and taking into account that $\chi_1(x) = -\chi_1(-x)$, $\chi_2(y) = -\chi_2(-y)$, we reduce problem (4.1) to the following system of two SIE with a fixed singularity at zero:

$$\frac{1}{\pi}\int_0^1 \frac{\omega_1(\tau)}{\tau - t}\,d\tau + \frac{1}{\pi}\int_0^1 \omega_2(\tau)\frac{\tau - \mu t}{(\tau + \mu t)^2}\,d\tau = h_1(t) \tag{4.2}$$

$$\frac{1}{\pi}\int_0^1 \omega_1(\tau)\frac{\tau - \mu^{-1}t}{(\tau + \mu^{-1}t)^2}\,d\tau + \frac{1}{\pi}\int_0^1 \frac{\omega_2(\tau)}{\tau - t}\,d\tau = h_2(t) \quad (0 < t < 1)$$

$$\omega_1(\tau) = \chi_1(a\sqrt{\tau}), \quad \omega_2(\tau) = \chi_2(b\sqrt{\tau})$$

$$h_1(t) = -4cq(a\sqrt{t}), \quad h_2(t) = -4cr(b\sqrt{t}), \quad \mu = (a/b)^2$$

According to the scheme of Sect.1, system (4.2) reduces to the Riemann problem (1.11) with the matrix coefficient

$$G(s) = \begin{Vmatrix} \rho_1(s) & \mu^{-s-1}\rho_2(s) \\ \mu^{s+1}\rho_2(s) & \rho_1(s) \end{Vmatrix}, \quad \begin{matrix} \rho_1(s) = \operatorname{ctg}\pi s \\ \rho_2(s) = (2s+1)\operatorname{cosec}\pi s \end{matrix} \tag{4.3}$$

We will represent $G(s)$ in the form $G(s) = G_+(s)\,G_0(s)\,G_-(s)$, where

$$G_0(s) = \begin{Vmatrix} [\rho_2(s) - \rho_1(s)]\,\mu^{s+1} & 0 \\ \rho_1(s) & [\rho_2(s) + \rho_1(s)]\,\mu^{-s-1} \end{Vmatrix}$$

and $G_\pm(s)$ are determined by (2.1) and we latter follow the scheme in Sect.2.

The stress intensity coefficients $\sigma_x(\pm 0, y)$ and $\sigma_y(x, \pm 0)$ at the slit vertices

$$K_a = \lim_{x \to a+0} \sqrt{2\pi(x-a)}\,\sigma_y(x, \pm 0),$$

$$K_b = \lim_{y \to b+0} \sqrt{2\pi(y-b)}\,\sigma_x(\pm 0, y)$$

are determined by the same method as in Sect.3, and have the form

$$K_a = -(\sqrt{2}c)^{-1}\sqrt{a}C_2, \quad K_b = (4\sqrt{2}c)^{-1}\sqrt{b}g_2^\circ(\mu^{-1})$$

We apply the proposed method to the problem of the longitudinal shear of an elastic plane with a defect V which is a pair of slits or inclusions converging at an arbitrary angle

$$\{(r, \theta) \in \mathbb{R}^2\colon 0 < r < a_1, \theta = +0, 2\pi - 0;\ 0 < r < a_2, \quad \theta_2 = \alpha \pm 0\}$$

The problem reduces to the vector problem (1.11) with a coefficient of the form (4.3) and then to problems 1° and 2° of (2.5). The method allows of extension to the case of a plane problem for an elastic plane with the defect V.

In conclusion we note that the problems of Sects.1-4, posed for bounded domains, reduce to SIE systems of the form

$$S\omega(t) + R\omega(t) = h(t), \quad 0 < t < 1 \tag{4.4}$$

where S is a two-dimensional singular integral operator with fixed singularity corresponding to the characteristic systems (1.7) or (4.2) and R is a regular operator $\omega = \|\omega_1, \omega_2\|$, $h = \|h_1, h_2\|$. The method elucidated in Sect.2 enables the SIE system (4.4) to be regularized by the Carleman-Vekua method to reduce it to a system of three Fredholm integral equations of the second kind.

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STATE OF THERMAL STRESS AND STRAIN OF A PLATE WEAKENED BY A RECTANGULAR HOLE*

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By using the method of continuation of functions a solution is obtained for the stationary heat conduction problem and for the corresponding static problem of thermo-elasticity for an infinite plate weakened by a rectangular hole.

1. Solution of the heat conduction problem. Let us consider a homogeneous isotropic unbounded plate of thickness 2δ with a rectangular cutout $|x_i| < a_i$ $(i = 1, 2)$. Heat transfer from the external medium occurs by Newton's law through the surface of the cutout and the side surfaces $x_3 = \pm\delta$. We ensure the temperature of the medium flowing over the surfaces $x_3 = \pm\delta$ to be zero, while the temperature of the medium flowing over the plate rectangular boundary is t_c. We then have the third boundary value problem for the Helmholtz equation in the domain external to the rectangle /1/ to determine the stationary temperature field T in the plate. We use the method of continuation of functions /2/ to solve this problem. To do this we introduce a new unknown function Θ that agrees with the desired function of the temperature T outside the rectangle and equals zero within, i.e.,

$$\Theta = TM(x_1, x_2) \tag{1.1}$$
$$M(x_1, x_2) = 1 - M(x_1)M(x_2), \quad M(x_i) = S_+(x_i + a_i) - S_-(x_i - a_i)$$
$$S_\pm(\xi) = \begin{cases} 1, & \xi > 0 \\ 0{,}5 \mp 0{,}5, & \xi = 0 \\ 0, & \xi < 0 \end{cases}$$

Taking account of the symmetry of the problem relative to the coordinate axes and the boundary conditions on the rectangle contour, we obtain an equation with singular coefficients for the function

$$\frac{\partial^2\Theta}{\partial x_1^2} + \frac{\partial^2\Theta}{\partial x_2^2} - \varkappa^2\Theta = \sum_{i=1}^{2} \{h_i (T|_{x_i=a_i} - t_c) M(x_{i\pm1}) \times [\delta_+(x_i + a_i) + \delta_-(x_i - a_i)] - T|_{x_i=a_i} M(x_{i\pm1}) \times [\delta_+'(x_i + a_i) - \delta_-'(x_i - a_i)]\} \tag{1.2}$$
$$h_i = \frac{\alpha_i}{\lambda}, \quad \varkappa^2 = \frac{\alpha_3}{\lambda\delta}, \quad i \pm 1 = \begin{cases} 2, & i = 1 \\ 1, & i = 2 \end{cases}$$

(λ is the thermal conductivity, α_3 and α_i $(i = 1, 2)$ are heat transfer coefficients from the surfaces $x_3 = \pm\delta$, and $|x_i| < a_i$, $|x_{i\pm1}| = a_{i\pm1}$).

The values of the function T on the rectangle contour that are in (1.2) are expanded in a Fourier series

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